

# QUARTIC FORMS IN MANY VARIABLES

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ABSTRACT. We show that a quartic  $p$ -adic form with at least 3192 variables possesses a non-trivial zero. We also prove new results on systems of cubic, quadratic and linear forms. As an example, we show that for a system comprising two cubic forms 132 variables are sufficient.

## 1. INTRODUCTION

Let  $p$  be a rational prime and  $F_1, \dots, F_r \in \mathbb{Q}_p[x_1, \dots, x_n]$  be forms with respective degrees  $d_1, \dots, d_r$ . E. Artin conjectured in the 1930s, that  $F_1, \dots, F_r$  have a common non-trivial zero provided

$$n > d_1^2 + \dots + d_r^2.$$

Unfortunately, this has been verified merely for a single quadratic (Hasse [6]), a single cubic (Lewis [8]) and a system comprising two quadratic forms (Demyanov [4] and independently Birch, Lewis and Murphy [1]). In fact counterexamples are known for many  $(d_1, \dots, d_r)$ . Although false in general Bauer [2] has shown there is a finite non-negative integer  $v(d_1, \dots, d_r)$ , independent of  $p$ , such that  $F_1, \dots, F_r$  possess a non-trivial zero whenever

$$n > v(d_1, \dots, d_r).$$

His proof reduces the problem to diagonal forms, which have been studied extensively (see in particular [3]). Refined subsequent results use quasi-diagonalisation techniques. The best general bound is due to Wooley [10]. For a system comprising  $r$  forms of degree  $d$  he showed that  $n > (rd^2)^{2^{d-1}}$  suffices. For a number of degrees better bounds are available. Firstly, we can extract better estimates from Wooley's proof for specific  $d$ . Secondly, Heath-Brown [7] considerably improved these for a single quartic by establishing  $v(4) \leq 4220$ . His proof has been adapted by Zahid [11] to show  $v(5) \leq 4562911$  (Note that in this case the conjecture has been confirmed if  $p > 7$ . See [5]). Heath-Brown's method provides better results if the involved degrees are not multiples

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of  $p$ . The purpose of this paper is to develop a variant yielding improved bounds if  $p$  does divide the degree.

**Theorem 1.**  $v(4) \leq 3191$

On the other hand, Terjanian [9] has constructed a dyadic quartic in 20 variables which lacks a non-trivial zero. Several results address specific systems of forms. To put the next result into perspective it suffices to know that  $v(3, 3) \leq 213$  can be derived by combining [7] and [11].

**Theorem 2.**  $v(3, 3) \leq 131$

The proofs of Theorem 1 and 2 rely on results for systems comprising a number of quadratic forms. These allow us to impose certain constraints on the shape of the forms involved. By applying Hensel's Lemma we then establish a non-trivial zero. Both theorems would enormously profit from better bounds on systems of quadratics. The method can be readily adapted for other degrees.

## 2. PRELIMINARIES

We introduce a few Lemmas we shall need in due course. For ease of notation we write  $V(r_3, r_2, r_1; p)$  for the least integer such that every system comprising  $r_3$  cubic,  $r_2$  quadratic and  $r_1$  linear  $p$ -adic forms possesses a non-trivial zero as soon as  $n > V(r_3, r_2, r_1; p)$ . Concerning quadratics the following bounds as found in [7] will be enough.

**Lemma 1.**

$$V(0, r, 0; p) \leq \begin{cases} 2r^2 - 16 & \text{if } r \geq 6 \text{ is even} \\ 2r^2 - 14 & \text{if } r \geq 7 \text{ is odd} \end{cases}$$

An estimate particular efficient for systems with just one cubic is due to Zahid [11].

**Lemma 2.** *Suppose  $p \neq 3$  and  $r_3 \geq 1$ . Then*

$$V(r_3, r_2, r_1; p) \leq V(r_3 - 1, 6(r_3 - 1) + r_2, 9r_3 + 6r_2 + r_1; p).$$

We shall later need to establish that certain vectors are linearly independent. Heath-Brown [7] provides an adequate criterion.

**Lemma 3.** *Let  $F \in \mathbb{Q}_p[x_1, \dots, x_n]$  be a form of degree  $d$ , having only the trivial zero in  $\mathbb{Q}_p$ . Let  $\mathbf{e}_1, \dots, \mathbf{e}_k$  be linearly independent vectors in  $\mathbb{Q}_p^n$ , and suppose that we have a non-zero vector  $\mathbf{e} \in \mathbb{Q}_p^n$  such that the form*

$$F_0(t_1, \dots, t_k, t) := F(t_1\mathbf{e}_1 + \dots + t_k\mathbf{e}_k + t\mathbf{e}),$$

in the indeterminates  $t_1, \dots, t_k$  and  $t$ , contains no terms of degree one in  $t$ . Then the set  $\{\mathbf{e}_1, \dots, \mathbf{e}_k, \mathbf{e}\}$  is linearly independent.

We denote the  $p$ -adic valuation of  $x \in \mathbb{Q}_p$  by  $\nu(x)$ . The following non-standard variant of Hensel's Lemma is crucial to our proofs.

**Lemma 4.** *Let  $f$  be a polynomial over  $\mathbb{Z}_p$  and suppose there exists an integer  $x$  such that  $\nu(f(x)) \geq \nu(f'(x))^2$  and, if equality holds, in addition  $\nu(f''(x)/2) \geq 1$ . Then there exists a  $p$ -adic integer  $y$  such that  $f(y) = 0$  and  $y = x \pmod{p}$ .*

### 3. PROOF OF THEOREM 1

It is sufficient to establish the dyadic case, since Heath-Brown has shown that 313 variables are enough if  $p$  is odd. As a first step we reduce this to a problem for a system of cubic, quadratic and linear forms.

**Lemma 5.**  $v(4) \leq \max\{V(4, 10, 20; 2), V(3, 18, 56; 2)\}$

*Proof of Lemma 5.* Let  $F \in \mathbb{Q}_2[x_1, \dots, x_n]$  be a quartic form. Suppose that  $F$  does have the trivial zero only and

$$(1) \quad n > \max\{V(4, 10, 20; 2), V(3, 18, 56; 2)\}.$$

We shall construct a subspace of  $\mathbb{Q}_2^n$  on which  $F$  is of special shape. By applying Hensel's Lemma we then find a non-trivial zero. In order to see how we can manipulate the shape assume that  $\mathbf{e}_1, \dots, \mathbf{e}_{k-1} \in \mathbb{Q}_2^n$  are linearly independent. If  $\mathbf{e}$  is an additional vector we write

$$F(x_1\mathbf{e}_1 + \dots + x_k\mathbf{e}_k + x\mathbf{e}) = F(x_1\mathbf{e}_1 + \dots + x_k\mathbf{e}_k) + \sum_{\sum d_i=3} \mathbf{x}^{\mathbf{d}} L_{\mathbf{d}}(\mathbf{e})x + \sum_{\sum d_i=2} \mathbf{x}^{\mathbf{d}} Q_{\mathbf{d}}(\mathbf{e})x^2 + \sum_{\sum d_i=1} \mathbf{x}^{\mathbf{d}} C_{\mathbf{d}}(\mathbf{e})x^3 + F(\mathbf{e})x^4,$$

where  $L_{\mathbf{d}}$  are linear,  $Q_{\mathbf{d}}$  quadratic and  $C_{\mathbf{d}}$  cubic forms. If we want that some of these forms (and respective monomials) vanish, they must have  $\mathbf{e}$  as a common non-trivial zero. This can be ensured at the cost of a condition on  $n$ . If, in particular,  $L_{\mathbf{d}}(\mathbf{e}) = 0$  for all  $\mathbf{d}$  such that  $\sum d_i = 1$ , then  $\mathbf{e}_1, \dots, \mathbf{e}_{k-1}, \mathbf{e}$  are by Lemma 3 linearly independent.

Thus we can successively choose vectors  $\mathbf{e}_1, \dots, \mathbf{e}_5$  such that

$$F(\mathbf{e}_1 x_1 + \dots + \mathbf{e}_5 x_5) = F(\mathbf{e}_1)x_1^4 + \dots + F(\mathbf{e}_5)x_5^4,$$

by imposing at most 4 cubic, 10 quadratic and 20 linear constraints (see (1)). Clearly,  $F(\mathbf{e}_i) \neq 0$  for all  $i$ . We show that we may assume

$$(2) \quad \nu(\mathbf{e}_1) = 0, \quad \nu(\mathbf{e}_2) = 1, \quad \nu(\mathbf{e}_3) = 2.$$

We say that a non-zero vector  $\mathbf{e} \in \mathbb{Q}_2^n$  has level  $r$  if  $\nu(F(\mathbf{e})) = r \pmod{4}$ . If  $\mathbf{e}_1, \dots, \mathbf{e}_5$  have three different levels, then (2) follows by relabelling and rescaling. Otherwise we can find three vectors  $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k$  of the same level  $r$  for some  $1 \leq i < j < k \leq 5$ . As we may assume that  $F(\mathbf{e}_i), F(\mathbf{e}_j), F(\mathbf{e}_k) \in \{-2^r, 2^r\}$ , there are  $s, t \in \{i, j, k\}$  such that  $F(\mathbf{e}_s) + F(\mathbf{e}_t) = \pm 2^{r+1} \pmod{2^{r+2}}$ . We replace  $\mathbf{e}_s$  and  $\mathbf{e}_t$  with  $\mathbf{e}'_s := \mathbf{e}_s + \mathbf{e}_t$ , which has level  $r+1 \pmod{4}$ , and a newly chosen vector  $\mathbf{e}'_t$ . If the resulting vectors still have two levels only, we repeat the argument until we obtain three vectors of different levels.

We choose new vectors  $\mathbf{e}_4$  of maximal level such that  $F(\mathbf{e}_1 x_1 + \dots + \mathbf{e}_4 x_4)$  is diagonal. We show that  $\mathbf{e}_4$  has level 2 at most. If  $\mathbf{e}_4$  has level 3, we choose a new vector  $\mathbf{e}_5$  such that  $F(\mathbf{e}_1 x_1 + \dots + \mathbf{e}_5 x_5)$  remains diagonal. By (2) we may, after rescaling and relabelling, assume that

$$(3) \quad \nu(\mathbf{e}_1) = 0, \quad \nu(\mathbf{e}_2) = 1, \quad \nu(\mathbf{e}_3) = 2, \quad \nu(\mathbf{e}_4) = 3, \quad \nu(\mathbf{e}_5) = 0.$$

Thus we can set  $x_1, x_5 = 1$  and pick  $x_i \in \{0, 1\}$  for  $2 \leq i \leq 4$  such that  $F(\mathbf{e}_1 x_1 + \dots + \mathbf{e}_5 x_5) = 0 \pmod{2^4}$ . The function  $f(t) := F(\mathbf{e}_1 t + \dots + \mathbf{e}_5 x_5)$  then satisfies  $\nu(f(1)) \geq 4$ ,  $\nu(f'(1)) = 2$  and  $\nu(f''(1)/2) \geq 1$ . By Hensel's Lemma  $F$  has a non-trivial zero, contrary to assumption.

We choose a new vector  $\mathbf{e}_5$  of maximal level such that

$$\begin{aligned} F(\mathbf{e}_1 x_1 + \dots + \mathbf{e}_5 x_5) &= F(\mathbf{e}_1) x_1^4 + F(\mathbf{e}_2) x_2^4 + F(\mathbf{e}_3) x_3^4 \\ &\quad + F(\mathbf{e}_4) x_4^4 + c_{45} x_4 x_5^3 + F(\mathbf{e}_5) x_5^4. \end{aligned}$$

By maximality  $\mathbf{e}_5$  can be of level 2 at most. We show that  $\nu(F(\mathbf{e}_5)) \leq 1$ . Suppose, after rescaling, that  $\nu(F(\mathbf{e}_4)), \nu(F(\mathbf{e}_5)) = 2$ . By a case-by-case analysis of  $\nu(c_{45})$  we establish a non-trivial zero. If  $\nu(c_{45}) < \nu(F(\mathbf{e}_5))$  we use Hensel's Lemma to lift  $\mathbf{e}_5$ . In the case of  $\nu(c_{45}) = 2$  it follows that  $2^{-2}F(\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5) = 0 \pmod{2}$ . Thus we can apply Hensel's Lemma to  $f(t) = 2^{-2}F(\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 t)$ . If  $\nu(c_{45}) = 3$  we may assume  $\nu(\mathbf{e}_4 + \mathbf{e}_5) > 3$ , since the case of four vectors of levels 0, 1, 2 and 3 has been discussed above (see (3)). Thus we can choose  $x_i \in \{0, 2\}$  such that  $F(\mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 + \mathbf{e}_3 x_3 + \mathbf{e}_4 + \mathbf{e}_5) = 0 \pmod{2^7}$ . Consequently, we can apply Hensel's Lemma to  $f(t) := F(\mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 + \mathbf{e}_3 x_3 + \mathbf{e}_4 t + \mathbf{e}_5)$ . The case  $\nu(c_{45}) = 4$  is slightly more involved. Since we may assume  $\nu(\mathbf{e}_4 + \mathbf{e}_5) > 3$ , it follows that  $F(\mathbf{e}_3) = F(\mathbf{e}_i) \pmod{2^4}$  for some  $i \in \{4, 5\}$ . Thus  $\mathbf{e}'_3 := \mathbf{e}_3 + \mathbf{e}_i$  is a vector such that  $\nu(\mathbf{e}'_3) = 3$ . Assume without loss of generality that  $i = 4$ . By (5) we can choose a new vector  $\mathbf{e}_6$  such that

$$\begin{aligned} F(\mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 + \mathbf{e}'_3 x_3 + \mathbf{e}_5 x_5 + \mathbf{e}_6 x_6) &= \\ F(\mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 + \mathbf{e}'_3 x_3 + \mathbf{e}_5 x_5) &+ F(\mathbf{e}_6) x_6^4. \end{aligned}$$

If  $\nu(\mathbf{e}_6) = 3$  then  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5, \mathbf{e}_6$  have levels 0, 1, 2 and 3, respectively, and we can proceed as in (3). The same works for  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}'_3, \mathbf{e}_6$  provided that  $\nu(\mathbf{e}_6) = 2$ . If  $\nu(\mathbf{e}_6) = 1$  we choose  $x_1, x_3, x_5 \in \{0, 1\}$  such that  $F(2\mathbf{e}_1x_1 + \mathbf{e}_2 + \mathbf{e}'_3x_3 + \mathbf{e}_5x_5 + \mathbf{e}_6) = 0 \pmod{2^5}$ . If we set  $f(t) = 2^{-1}F(2\mathbf{e}_1x_1 + \mathbf{e}_2 + \mathbf{e}'_3x_3 + \mathbf{e}_5x_5 + \mathbf{e}_6t)$  then  $\nu(f(1)) \geq 4$ ,  $\nu(f'(1)) = 2$  and  $\nu(f''(1)) \geq 1$  hold true and Hensel's Lemma can be applied. Similarly we can find  $x_2, x_3, x_5 \in \{0, 1\}$  such that  $F(\mathbf{e}_1 + \mathbf{e}_2x_2 + \mathbf{e}'_3x_3 + \mathbf{e}_5x_5 + \mathbf{e}_6) = 0 \pmod{2^4}$  provided  $\nu(\mathbf{e}_6) = 0$ . Consequently, Hensel's Lemma yields a non-trivial zero.

By (5) we can choose a new vector  $\mathbf{e}_6$  of maximal level such that

$$\begin{aligned} F(\mathbf{e}_1x_1 + \cdots + \mathbf{e}_6x_6) &= F(\mathbf{e}_1)x_1^4 + F(\mathbf{e}_2)x_2^4 + F(\mathbf{e}_3)x_3^4 + F(\mathbf{e}_4)x_4^4 \\ &\quad + c_{45}x_4x_5^3 + F(\mathbf{e}_5)x_5^4 + c_{46}x_4x_6^3 + c_{56}x_5x_6^3 + c_{456}x_4x_5x_6^2 + F(\mathbf{e}_6)x_6^4. \end{aligned}$$

By maximality  $\mathbf{e}_6$  has level 1 at most. Suppose after rescaling that  $\nu(\mathbf{e}_5), \nu(\mathbf{e}_6) = 1$ . If  $\nu(c_{56}) < 1$ , we can lift  $\mathbf{e}_6$  via Hensel's Lemma. If  $\nu(c_{56}) = 1$ , there are  $x_2, x_5, x_6 \in \{0, 1\}$  such that  $2^{-1}F(\mathbf{e}_2x_2 + \mathbf{e}_5x_5 + \mathbf{e}_6x_6) = 0 \pmod{2}$  and one of it's partial derivatives does not vanish modulo 2. Thus there exists a non-trivial zero and we may assume that  $\nu(c_{56}) \geq 2$ . By maximality  $\mathbf{e}_5 + \mathbf{e}_6$  can not have level 2 or 3. Hence we can find  $x_1, x_2 \in \{0, 2\}$  such that  $F(\mathbf{e}_1x_1 + \mathbf{e}_2x_2 + \mathbf{e}_5 + \mathbf{e}_6) = 0 \pmod{2^9}$ . Consequently, we can apply Hensel's Lemma to  $f(t) := F(\mathbf{e}_1x_1 + \mathbf{e}_2x_2 + \mathbf{e}_5t + \mathbf{e}_6)$ .

By (1) we can choose final vector  $\mathbf{e}_7$  of maximal level such that

$$\begin{aligned} F(\mathbf{e}_1x_1 + \cdots + \mathbf{e}_7x_7) &= F(\mathbf{e}_1)x_1^4 + F(\mathbf{e}_2)x_2^4 + F(\mathbf{e}_3)x_3^4 + F(\mathbf{e}_4)x_4^4 \\ &\quad + c_{45}x_4x_5^3 + F(\mathbf{e}_5)x_5^4 + c_{46}x_4x_6^3 + c_{56}x_5x_6^3 + c_{456}x_4x_5x_6^2 \\ &\quad + F(\mathbf{e}_6)x_6^4 + c_{47}x_4x_7^3 + c_{57}x_5x_7^3 + c_{67}x_6x_7^3 + c_{457}x_4x_5x_7^2 \\ &\quad + c_{467}x_4x_6x_7^2 + c_{567}x_5x_6x_7^2 + F(\mathbf{e}_7)x_7^4. \end{aligned}$$

By maximality  $\mathbf{e}_7$  has level 0. Suppose that  $\nu(\mathbf{e}_6), \nu(\mathbf{e}_7) = 0$ . If  $\nu(c_{67}) < 0$ , we lift  $\mathbf{e}_7$ . In case of  $\nu(c_{67}) = 0$  we can find  $x_1, x_2 \in \{0, 1\}$  such that  $F(\mathbf{e}_1x_1 + \mathbf{e}_6x_6 + \mathbf{e}_7x_7) = 0 \pmod{2}$  and Hensel's Lemma can be applied. Finally, suppose that  $\nu(c_{67}) > 0$ . Since  $\mathbf{e}_6 + \mathbf{e}_7$  can not have level 1, 2, 3 we can find  $x_1 \in \{0, 1\}$  such that  $F(\mathbf{e}_1x_1 + \mathbf{e}_6 + \mathbf{e}_7) = 0 \pmod{2^8}$ . Lifting this zero completes the proof of Lemma 5.  $\square$

In order to estimate the quantities of  $V(4, 10, 20; 2)$  and  $V(3, 18, 56; 2)$  we provide an improved estimate.

**Lemma 6.** *Suppose  $p \equiv 2 \pmod{3}$  and  $r_3 \geq 1$ . Then*

$$V(r_3, r_2, r_1; p) \leq V(r_3 - 1, 3r_3 + r_2, 3r_3 + 3r_2 + r_1; p).$$

Theorem 1 now easily follows from Lemmas 1, 2, 5 and 6.

*Proof of Lemma 6.* It is enough to show that  $V(r_3, r_2, 0; p) \leq V(r_3 - 1, 3r_3 + r_2, 3r_3 + 3r_2; p)$ . We choose a cubic form  $C$  and denote by  $\mathbf{G}$  the system comprising all other forms. Suppose that for all non-zero  $\mathbf{x} \in \mathbb{Q}_p$  such that  $\mathbf{G}(\mathbf{x}) = 0$  we have  $C(\mathbf{x}) \neq 0$ . Assume in addition that

$$(4) \quad n > V(r_3 - 1, 3r_3 + r_2, 3r_3 + 3r_2; p).$$

By (4) there exists  $\mathbf{e}_1$  such that  $C(\mathbf{e}_1 x_1) = C(\mathbf{e}_1) x_1^4$  and  $\mathbf{G}(\mathbf{e}_1 x_1)$  is identically zero. We shall successively choose further vectors. A non-zero vector  $\mathbf{e}$  is said to have level  $r \in \mathbb{F}_3$  if  $\nu(C(\mathbf{e})) = r \pmod{3}$ . Suppose we have chosen  $s$  vectors  $\mathbf{e}_1, \dots, \mathbf{e}_s$  of different levels such that  $C(\mathbf{e}_i x_i) = C(\mathbf{e}_i) x_i^4$  and  $\mathbf{G}(\mathbf{e}_i x_i) = 0$  for all  $1 \leq i \leq s$ . Since  $s \leq 3$  and (4) we can choose an additional vector  $\mathbf{e}_{s+1}$  such that

$$(5) \quad C(\mathbf{e}_i x_i + \mathbf{e}_{s+1} x_{s+1}) = C(\mathbf{e}_i) x_i^3 + C(\mathbf{e}_{s+1}) x_{s+1}^3$$

and  $\mathbf{G}(\mathbf{e}_i x_i + \mathbf{e}_{s+1} x_{s+1})$  is identically zero for all  $1 \leq i \leq s$ . It follows from Lemma 3 that  $\mathbf{e}_i$  and  $\mathbf{e}_{s+1}$  are linearly independent for each  $1 \leq i \leq s$ . By iterating this argument we find two vectors  $\mathbf{e}_i, \mathbf{e}_j$  of the same level for some  $1 \leq i < j \leq 4$  such that  $C(\mathbf{e}_i x_i + \mathbf{e}_j x_j)$  is diagonal. After rescaling both the variables and the form we may assume that  $\nu(C(\mathbf{e}_i)), \nu(C(\mathbf{e}_j)) = 0$ . Since  $p = 2 \pmod{3}$ , there exists  $t \in \mathbb{F}_p$  such that  $\nu(C(\mathbf{e}_i t + \mathbf{e}_j)) \geq 1$  and  $\nu(C'(\mathbf{e}_i t + \mathbf{e}_j)) = 0$ . The Lemma then follows by applying Hensel's Lemma.  $\square$

#### 4. PROOF OF THEOREM 2

We crucially establish a new bound if  $p = 3$ .

**Lemma 7.** *Suppose  $r_3 \geq 1$ . Then*

$$V(r_3, r_2, r_1; 3) \leq V(r_3 - 1, 3r_3 + r_2, 6r_3 + 3r_2 + r_1; 3).$$

Theorem 2 now follows in conjunction with Lemmas 1 and 2. Also note the improvement provided by Lemma 6 if  $p = 2 \pmod{3}$ .

*Proof.* It suffices to prove that  $V(r_3, r_2, 0; 3) \leq V(r_3 - 1, 3r_3 + r_2, 6r_3 + 3r_2; 3)$ . We choose a cubic form  $C$  and denote by  $\mathbf{G}$  the system comprising all other forms. Suppose that for all non-zero  $\mathbf{x} \in \mathbb{Q}_3$  such that  $\mathbf{G}(\mathbf{x}) = 0$  we have  $C(\mathbf{x}) \neq 0$ . Assume in addition that

$$(6) \quad n > V(r_3 - 1, 3r_3 + r_2, 6r_3 + 3r_2; 3).$$

By (6) we can successively choose non-zero vectors  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$  such that

$$(7) \quad C(\mathbf{e}_1 x_1 + \dots + \mathbf{e}_4 x_4) = C(\mathbf{e}_1) x_1^3 + C(\mathbf{e}_2) x_2^3 + C(\mathbf{e}_3) x_3^3 + C(\mathbf{e}_4) x_4^3$$

and  $\mathbf{G}(\mathbf{e}_1x_1 + \cdots + \mathbf{e}_4x_4)$  is identical zero. By Lemma 3 are  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  and  $\mathbf{e}_4$  linearly independent. A vector  $\mathbf{e} \in \mathbb{Q}_3 - 0$  is said to have level  $r \in \mathbb{F}_3$  if  $\nu(C(\mathbf{e})) = r \pmod{3}$ . Suppose there are two vectors  $\mathbf{e}_i, \mathbf{e}_j$  of the same level for some  $1 \leq i < j \leq 4$ . We rescale both the variables and the form such that  $\nu(\mathbf{e}_i), \nu(\mathbf{e}_j) = 0$ . Since  $C(\mathbf{e}_i), C(\mathbf{e}_j) = \pm 1 \pmod{3}$  there exists  $t_0 \in \{1, -1\}$  such that  $3 \mid C(\mathbf{e}_it_0 + \mathbf{e}_j)$ . If we set  $f(t) = C(\mathbf{e}_it + \mathbf{e}_j)$  either  $\nu(f(t_0)) \geq 2$ ,  $\nu(f'(t_0)) = 1$  and  $\nu(f''(t_0)) \geq 1$  such that Hensel's Lemma can be applied or  $\mathbf{e}'_i := \mathbf{e}_it_0 + \mathbf{e}_j$  is of level 1. Thus we can replace three vectors of the same level  $r$  by two of level  $r$  and  $r+1$ . We then choose an additional fourth vector. By repeating this argument, relabelling and rescaling we find vectors  $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k$  for some  $1 \leq i < j < k \leq 4$  such that  $\nu(\mathbf{e}_i), \nu(\mathbf{e}_j) = 0$  and  $\nu(\mathbf{e}_k) = 1$ . We set  $x_j = -C(\mathbf{e}_i)$  such that  $3 \mid C(\mathbf{e}_i + \mathbf{e}_jx_j)$ . Thus we can write  $C(\mathbf{e}_i + \mathbf{e}_jx_j) = 3s$  and  $C(\mathbf{e}_k) = 3l$  where  $3 \nmid l$ . We set  $x_3 = -sl$  if  $s$  is a  $p$ -adic unit and  $x_3 = 0$  otherwise. If we write  $f(t) = C(\mathbf{e}_it + \mathbf{e}_jx_j + \mathbf{e}_kx_k)$ , then  $\nu(f(1)) \geq 2$ ,  $\nu(f'(1)) = 1$ ,  $\nu(f''(1)) \geq 1$  and Hensel's Lemma applies.  $\square$

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